MODEL DESCRIPTION OF ROLL-WAVES

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The equation considered here in which a small parameter accompanies the higher derivative (the vanishing-viscosity type equation) differs from the model turbulence equation due to Burgers [1] in that the solutions of the corresponding degenerate equation (at zero viscosity) possess a periodic dependence on a linear combination of the coordinate and time only, and satisfy the conditions of the type of conservation and dissipation law at the discontinuities. We investigate how such solutions of the degenerate equation may be approximated by smooth solutions of an equation containing a vanishing viscosity term, the smooth solutions depending on the same combination of the independent variables. These solutions and their passage to the limit are of interest when constructing a mathematical description of the roll-waves.

Under certain, not yet fully investigated conditions determined by the channel parameters and the character of the perturbing force, so-called roll-waves (henceforth to be denoted by RW) appear in the open, steady state, turbulent flows. These waves show a sharply defined periodicity with appreciable concentration of the fluid mass taking place within narrow zones situated near the wavefronts moving with constant velocity. This phenomenon may lead to the necessity of reducing the rate of flow in the channel and may also generate sharp pulsating stresses in the streamlined surfaces [2 and 3].

As the RW are periodic and have constant velocity, self-similar periodic solutions $U(y, t)$ of the equations of hydrodynamics must be used to provide their mathematical description, the solutions depending only on the wave argument x

$$U(y, t) = u(x), \qquad x = y - ct \tag{0.1}$$

Here y is the coordinate parallel to the direction of motion of the RW, c is the velocity of the RW, t is time and $u(x)$ denotes the profile of the RW.

Only the self-similar solutions of the type (0.1) will be considered. Those which have a smooth or a piece-wise smooth profile and for which the relations of the type of the well known hydrodynamic conditions of conservation and dissipation hold on each discontinuity, shall be called the RW solutions (a more accurate definition is given below in Sect. 1).

The present paper deals with a model description of the roll-waves using the RW solutions of

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial y}\left(\frac{1}{2}u^2\right) = d(u - c) + \varepsilon\frac{\partial^2 u}{\partial y^2}; \quad -\infty < d < \infty, \quad c > 0, \quad \varepsilon \geqslant 0 \tag{0.2}$$

This equation differs from the well known model turbulance equation due to Burgers [1] by an additional term $d(u - c)$ (terms of this type are called the turbulent drag terms and appear in the dynamic equation of the St. Venant system of hydraulic equations widely used in computing nonsteady turbulent flows [4]).

When the last term which describes the influence of viscosity is deleted from (0.2), the resulting expression can be regarded as a model equation for the St. Venant system (like the latter system this equation has no smooth RW solutions, although it has piecewise smooth RW solutions [2]). Inclusion of the dissipative term $\varepsilon\partial^2 u / \partial y^2$ into (0.2) removes sharp peaks and vertical fronts from the RW solution profiles. It will be shown later that the "blurring" of these fronts can be reduced as much as we please by making $\varepsilon > 0$ sufficiently small.

We note that if we put $d = 0$ in (0.2), i.e. if we restrict ourselves to the Burgers equation, we shall not be able to approximate the vertical front of the RW solutions of the equation with "zero" viscosity, using the curved fronts of the smooth RW solutions of the equation with "vanishing" viscosity. The reason for this is, that at zero viscosity the Burgers equation has neither smooth, nor piecewise smooth RW solutions.

A sequence of RW solutions of (0.2) which converge as $\varepsilon \to 0$ can be constructed by choosing for each value of $\varepsilon > 0$ that RW solution, for which a certain functional assumes the same value as for the approximated RW solution of this equation obtained when $\varepsilon = 0$.

In Sect. 2 a wavelength type functional is used to construct a convergent sequence of the RW solutions of (0.2). In Sect. 3 we obtain the principal term of the asymptotic expression describing the deviation, with $\varepsilon \to 0$ of the RW solution of (0.2) selected with the aid of this functional, from the RW solution of (0.2) for $\varepsilon = 0$ (the rate of convergence as viscosity tends to zero). This deviation is averaged, with the exponent $p \geqslant 1$ over a region on the yt-plane, which may be arbitrarily large and contain discontinuities in the RW solutions of (0.2) at $\varepsilon = 0$. In the Sect.4 and 5 similar results are also obtained for the amplitude functional of the RW solution.

Asymptotic estimates of this kind find use in quantitative descriptions of the dissipative terms such as $\varepsilon\partial^2 u / \partial y^2$. Moreover, such estimates help to judge the accuracy of finite difference schemes which use the "numerical" iscosity to approximate to the first order quasi-linear differential equations [5] encountered in hydromechanics.

Inequalities providing an estimate of certain norms of the differences between the non-self-similar solutions of the degenerate and non-degenerate equations were proved in [6 - 8].

In Sect. 6 the passage to the limit is interpreted on the phase plane of the profile equation corresponding to (0.2). The problem of the passage to the limit, with vanishing viscosity in the class of solutions of the form (0.1) of a quasi-linear parabolic equation, was formulated in [9]. In [10] it was shown that if such a passage to the limit with $\varepsilon \to 0$ is possible, then it is possible (provided that the term characterizing the turbulent drag has well defined properties) only in the RW solution of the initial hyperbolic equation containing one discontinuity per period of the profile. Below we show that each RW solution of (0.2) obtained for $\varepsilon = 0$ and containing one discontinuity per period of the profile is indeed the limit as $\varepsilon \to 0$ of the corresponding sequence of the RW solutions of (0.2) with $\varepsilon > 0$.

1. The solutions under consideration and the method of approximation. The RW solution of the equation

$$\frac{\partial U_\varepsilon(y,t)}{\partial t} + \frac{\partial}{\partial y}\left(\frac{1}{2}U_\varepsilon^2(y,t)\right) = U_\varepsilon(y,t) - c + \varepsilon\frac{\partial^2 U_\varepsilon(y,t)}{\partial y^2} \quad (\varepsilon, c > 0) \quad (1.1)$$

shall be called its classical solution of the form $U_\varepsilon(y,t) = u_\varepsilon(x)$ bounded uniformly on $-\infty < x < \infty$ (these solutions will be denoted by u_ε).

The RW solution of the equation

$$\frac{\partial U_0(y,t)}{\partial t} + \frac{\partial}{\partial y}\left(\frac{1}{2}U_0^2(y,t)\right) = U_0(y,t) - c \quad (1.2)$$

shall be called the function of the form $U_0(y,t) = u_0(x)$ with the profile $u_0(x)$ periodic in x and containing not more than a finite number of the first order discontinuities x_i per period. When $x \neq x_i$, this function is continuously differentiable and satisfies (1.2), and the limit values attained on approaching the discontinuities from the left and from the right, satisfy the relations [2]

$$u_0(x_i+0) + u_0(x_i-0) = 2c, \quad u_0(x_i+0) < u_0(x_i-0) \quad (i = 1, 2, \ldots) \quad (1.3)$$

(We shall denote such solutions of (1.2) by u_0.

Relations (1.3) at the discontinuities of u_0 represent the Hugoniot and dissipation relations at the discontinuity corresponding to (1.2).

The set of all u_ε corresponding to the given $\varepsilon \geqslant 0$ and assuming its minimum value at $x = 0$, shall be denoted by M_ε. The remaining u_ε differ from those entering M_ε only by an insignificant shift in the value of the argument x of the profile $u_\varepsilon(x)$, $\varepsilon \geqslant 0$. We shall consider the approach of an arbitrarily fixed solution $U_0(y,t) \in M_0$ by the solutions $U_\varepsilon(y,t) \in M_\varepsilon$, $\varepsilon > 0$.

We note that by virtue of the theorems on vanishing viscosity [5] the sequence of solutions $U_\varepsilon(y,t)$ of (1.1) converges to $U_0(y,t)$ as $\varepsilon \to 0$ provided that the initial conditions are equal to each other, i.e.

$$U_\varepsilon(y,t)|_{t=0} = U_0(y,t)|_{t=0} \qquad (-\infty < y < \infty, \ \varepsilon > 0)$$

However, the elements of a sequence constructed with the help of the above condition are not functions of the wave argument $x = y - ct$ only. Thus in this case the piecewise smooth RW solutions of (1.2) are not approximated in a manner, natural in the physical sense, by the smooth solutions of (1.1) of similar structure and containing the viscosity term To obtain the required approximation $u_\varepsilon \to u_0$ as $\varepsilon \to 0$, we shall consider a different method of selecting the elements of the convergent sequence u_ε for each value of $\varepsilon > 0$

We shall approach an arbitrarily fixed solution $U_0(y,t) \in M_0$ by choosing, for each $\varepsilon > 0$ a solution $U_\varepsilon(y,t) \in M_\varepsilon$ satisfying the condition

$$\Phi(U_\varepsilon(y,t)) = \Phi(U_0(y,t)), \qquad \varepsilon > 0 \quad (1.4)$$

Here Φ is a functional with the following properties:

1. $D(\Phi) \supseteq \bigcup\limits_{\varepsilon \geqslant 0} M_\varepsilon$.

2. For each solution $U_0(y,t) \in M_0$ a number $\varepsilon(U_0, \Phi)$ exists such that for $0 < \varepsilon < \varepsilon(U_0, \Phi)$ a unique solution $U_\varepsilon(y,t)$ satisfying (1.4) exists in M_ε

3. The sequence $U_\varepsilon(y,t)$ determined by (1.4) satisfies

$$\| U_0(y,t) - U_\varepsilon(y,t) \|_{L_p(\Omega)} \to 0, \ \ \varepsilon \to 0 \quad (p \geqslant 1, \ \operatorname{mes} \Omega < \infty) \tag{1.5}$$

The class of the functionals of the type Φ is not empty; the functionals T and A considered in Sect. 2 and 4 respectively also have the Properties 1 to 3. We find, more accurately, how they converge in the vicinity and away from the discontinuities in u_0 In particular, the principal term of the asymptotics of the left-hand part of (1.5), is obtained for $\varepsilon \to 0$

From (1.1) we obtain for each fixed $\varepsilon > 0$

$$M_\varepsilon = \{U_\varepsilon(y,t,C) \mid U_\varepsilon(y,t,C) = u_\varepsilon(x,\bar{C}), \ 1 \leqslant C < \infty\}$$

Here $u_\varepsilon(x, C)$ is determined by a one-parameter C -family $(u_\varepsilon(x, C), v_\varepsilon(x, C))$ of bounded solutions of the autonomous system

$$\varepsilon \frac{dv}{dx} = (1-v)(c-u), \qquad \frac{du}{dx} = v \tag{1.6}$$

The bounded solutions of this system have corresponding cycles $L(\varepsilon, C)$ on the uv phase plane, determined by the equation

$$\frac{(u-c)^2}{2\varepsilon} - (f(v) + \ln C) = 0 \qquad (f(v) = \ln(1-v) + v, \ \ 1 \leqslant C < \infty) \tag{1.7}$$

The cycles are symmetrical with respect to the straight line $u = c$ and fill the half-plane $v < 1$. The other half-plane $v \geqslant 1$ is filled with unbounded trajectories which shall not be considered here in view of the definition of the RW solutions given above.

Setting $\varepsilon = 0$ in (1.6) we find that v_0 containing any prescribed number of the points of discontinuity per period of the profile may be found in M_0. Nevertheless, only those profiles u_0 in which two neighboring points of discontinuity are separated by the length of the period [10] can be regarded as limiting for the sequence of profiles u_ε with $\varepsilon \to 0$. Such profiles u_0 have the same left (right) limit values for all points of discontinuity.

Taking (1.3) into account we find M_0 represented by the following family containing a parameter $u^+ (-\infty < u^+ < c)$:

$$M_0 = \{U_0(y, t, u^+) \mid U_0(y, t, u^+) = u_0(x, u^+); \ u_0(x, u^+) = u^+ + x \ \text{ for } \ 0 \leqslant x < u^- - u^+, \ u_0(x + u^- - u^+, u^+) = u_0(x, u^+) \ \text{ for } \ -\infty < x < \infty\}$$
$$(u^- = 2c - u^+) \tag{1.8}$$

The dependence of the functions belonging to M_ε (M_0) and of their profiles and derivatives on the parameter C (or respectively u^+) will not be given each time.

2. The convergence when the periods u_0 and u_ε are equal. Let us define the functional T on $\bigcup_{\varepsilon \geqslant 0} M_\varepsilon$ by putting each solution belonging to M_ε, $\varepsilon \geqslant 0$ into one-to-one correspondence with the period of its profile $u_\varepsilon(x)$.

From (1.8) follows:

$$T(U_0(y,t,u^+)) \to 0 \quad \text{for} \quad u^+ \to c - 0, \quad T(U_0(y,t,u^+)) \to \infty \quad \text{for} \quad u^+ \to -\infty$$

while from (1.6) and (1.7) we have [11]

$$T(U_\varepsilon(y,t,C)) = T^+(\varepsilon, C) + T^-(\varepsilon, C), \quad T^+(\varepsilon, C) = \sqrt{2\varepsilon} \int_0^1 \varphi(z, q_\varepsilon(C))\, dz$$

$$T^-(\varepsilon, C) = \sqrt{2\varepsilon}\int_0^1 \varphi(z, -w_\varepsilon(C))\,dz, \quad \varphi(z, q) = q\,(f(zq) + \ln C)^{-1/2}(1 - zq)^{-1}$$

$$1 > q_\varepsilon(C) = \max_{(u,v)\in L(\varepsilon, C)} v \to 1, \quad 0 < w_\varepsilon(C) = -\min_{(u,v)\in L(\varepsilon, C)} v \to \infty$$

$$T(U_\varepsilon(y, t, C)) \to \infty \quad \text{for} \quad C \to \infty, \quad \varepsilon = \text{const}$$

$$q_\varepsilon(C) \to 0, \ T(U_\varepsilon(y, t, C)) \to 2\sqrt{\varepsilon}\,\pi \quad \text{for} \quad C \to 1 + 0, \ \varepsilon = \text{const}$$

$$f(q_\varepsilon(C)) = f(-w_\varepsilon(C)) = -\ln C, \frac{\partial q_\varepsilon(C)}{\partial C} > 0 \quad \text{for} \quad 1 < C < \infty, \quad \varepsilon > 0 \qquad (2.1)$$

Passing in (1.6) to polar coordinates and using a theorem on periods [12] we find, that when $\varepsilon = \text{const}$, $T(U_\varepsilon(y, t, C))$ is strictly monotonous and increases with C, $1 < C < \infty$, assuming all values between $2\sqrt{\varepsilon}\pi$ and ∞. Thus, Eq. (1.4) containing the functional T as Φ has a unique solution $U_\varepsilon(y, t) \in M_\varepsilon$ for any $U_0(y, t) \in M_0$ and any fixed value of ε, $0 < \varepsilon < \varepsilon(U_0, T) = \pi^{-2}(c - u^+)^2$.

The functional T also possesses the Property 3 of Sect. 1. Indeed, let us fix any $U_0(y, t) \in M_0$. We denote by C_ε the value of the parameter C of that function belonging to M_ε, which satisfies (1.4) when $T = \Phi$ and $U_0(y, t)$ is substituted into its right-hand side. The dependence of C_ε on u^+ is not given here. From (1.8) and (2.1) follows:

$$C_\varepsilon \to \infty, \ q_\varepsilon(C_\varepsilon) \to 1, \ w_\varepsilon(C_\varepsilon) \mapsto \infty, \ T^-(\varepsilon, C_\varepsilon) \to 0$$
$$\min_{(u,v)\in L(\varepsilon, C_\varepsilon)} u \to u^+, \quad \max_{(u,v)\in L(\varepsilon, C_\varepsilon)} u \to u^- \quad \text{for} \ \varepsilon \to 0 \qquad (2.2)$$

Moreover, taking into account the fact that the function $u_\varepsilon(x, C) - u_0(x, u^+)$ decreases monotonously in $x \in (0, 1)$ we find that when ε, C and u^+ are fixed, and condition (1.4) in which $\Phi = T$ holds, u_ε converges uniformly to u_0 with respect to the points on the xt-plane not belonging to the union of arbitrarily narrow strips adjacent to the lines of discontinuity of u_0

$$\| U_0(y, t, u^+) - U_\varepsilon(y, t, C_\varepsilon) \|_{C(\Pi_\delta)} \to 0 \quad \text{for} \ \varepsilon \to 0$$

$$\Pi_\delta = \{(y, t) \mid -\infty < y, t < \infty\} \setminus \bigcup_{-\infty<k<\infty} \{(y, t) \mid kT_0 - \delta < y - ct < kT_0\}$$

$$T_0 = T(U_0(y, t, u^+)) = u^- - u^+, \ -\infty < u^+ < c, \ 0 < \delta < T_0 \qquad (2.3)$$

Since δ can be made arbitrarily small, from (2.2) we can obtain (1.5).

3. The rate of convergence of $u_\varepsilon \to u_0$ when the periods are equal. We denote

$$q_\varepsilon = q_\varepsilon(C_\varepsilon), \ w_\varepsilon = w_\varepsilon(C_\varepsilon), \ \alpha_\varepsilon = {}^1\!/_2 T^+(\varepsilon, C_\varepsilon), \ \beta_\varepsilon = {}^1\!/_2 T^-(\varepsilon, C_\varepsilon)$$

$$J(\varepsilon, w, z, y) = (\varepsilon/2)^{1/2}\int_y^z (f(v) - f(w))^{-1/2}dv$$

and agree to connect two functions of ε with the sign $\approx$ if their quotient tends to unity when $\varepsilon \to 0$.

Performing the change of variables $y = \ln(1 - zq_\varepsilon)\ln^{-1}(1 - q_\varepsilon)$ and $y = \ln(1 + zw_\varepsilon)\ln^{-1}(1 + w_\varepsilon)$ in the integrals $T^+(\varepsilon, C_\varepsilon)$ and $T^-(\varepsilon, C_\varepsilon)$

respectively, we transform (1.4) with $\Phi = T$, taking into account (2.2), to the form [11]:

$$(2\varepsilon)^{-1/2} T_0 = \sqrt{-\ln(1-q_\varepsilon)}\, J^+(q_\varepsilon) + w_\varepsilon^{-1/2} \ln(1+w_\varepsilon) J^-(w_\varepsilon)$$

$$J^+(q_\varepsilon) \approx 2, \quad J^-(w_\varepsilon) \approx 1, \quad \frac{dJ^-(w_\varepsilon)}{dw_\varepsilon} = o(w_\varepsilon^{-1}) \tag{3.1}$$

$$\frac{dJ^+(q_\varepsilon)}{dq_\varepsilon} = (1-q_\varepsilon)^{-1} \ln(1-q_\varepsilon)\, o(1)$$

From (3.1), (2.1) and equations obtained by their termwise differentiation with respect to ε, we obtain, taking into account (2.2),

$$\ln(1-q_\varepsilon) \approx -{}^1/_8 T_0^2 \varepsilon^{-1}, \quad dw_\varepsilon/d\varepsilon \approx -{}^1/_8 T_0^2 \varepsilon^{-2}, \quad w_\varepsilon \approx {}^1/_8 T_0^2 \varepsilon^{-1} \tag{3.2}$$

$$\ln(1+w_\varepsilon) \approx -\ln\varepsilon, \quad \beta_\varepsilon \approx -2T_0^{-1}\varepsilon\ln\varepsilon, \quad J(\varepsilon, q_\varepsilon, q_\varepsilon, 0) = O(\varepsilon)$$

The penultimate relation in (3.2) provides an estimate of the distance along the x-axis on the xu-plane, by which the crest $(2\alpha_\varepsilon, u_\varepsilon(2\alpha_\varepsilon, C_\varepsilon))$ of the profile $u_\varepsilon \in M_\varepsilon$ fixed by the condition (1.4) with $\Phi = T$ on an arbitrary $u_0 \in M_0$ lags behind the crest (T_0, u^-) of this profile u_0. This lag is illustrated in Fig. 1. The acute-angled profile $u_0(x)$ of the RW solution u_0 is depicted by the thin line, and the profiles $u_\varepsilon(x)$ of the RW solution u_ε are shown by the lines whose thickness increases with increasing ε.

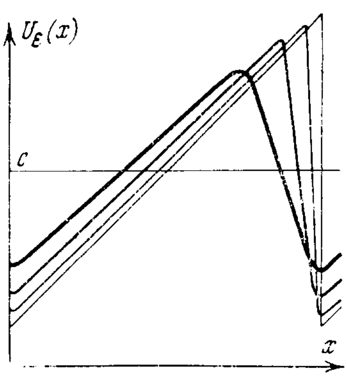

Fig. 1.

Let us now obtain the rates of convergence of (2.3) and (1.5) (these are given, respectively, by (3.4) and (3.9)).

From (1.6) to (1.8) and (2.1) we have

$$dx = \operatorname{sign}(c - u_\varepsilon(x)) \left(\frac{\varepsilon}{2}\right)^{1/2} (1-v)^{-1} (f(v) - f(q_\varepsilon))^{-1/2} dv$$

$$u_\varepsilon(x) = c - \int_x^{\alpha_\varepsilon} v_\varepsilon(x)\, dx, \qquad u_0(x) = c - \int_x^{\alpha_\varepsilon} dx - \beta_\varepsilon \tag{3.3}$$

$$u_0(x) - u_\varepsilon(x) = \operatorname{sign}(x - \alpha_\varepsilon) J(\varepsilon, q_\varepsilon, q_\varepsilon, v_\varepsilon(x)) - \beta_\varepsilon \quad (0 \leqslant x \leqslant 2\alpha_\varepsilon)$$

which, together with the last two relations of (3.2), show that $u_0(2\alpha_\varepsilon) - u_\varepsilon(2\alpha_\varepsilon) < 0$ for all sufficiently small ε. Consequently the point of intersection of the profiles u_ε and u_0 on the xu-plane lies to the right of the maximum $x = 2\alpha_\varepsilon$ of the profile u_ε (Fig. 1). Thus, by virtue of (1.8), (2.1), (2.2) and (3.3) we obtain the principal term of the norm in (2.3) with $\varepsilon \to 0$

$$\| U_0(y,t,u^+) - U_\varepsilon(y,t,C_\varepsilon) \|_{C(\Pi_\delta)} = u_\varepsilon(0, C_\varepsilon) - u^+ \approx 2T_0^{-1} |\varepsilon\ln\varepsilon| \tag{3.4}$$

The above relation gives an estimate of the deviation of u_ε from u_0 under the condition (1.4) with $\Phi = T$, in a uniform metric on the xt-plane with arbitrarily narrow neighborhoods of the lines of discontinuity of u_0 excluded.

To estimate the rates of convergence of $u_\varepsilon \to u_0$ in the regions lying on the xt -plane and containing the lines of discontinuity of u_0, we pass to the estimates in L_p. From (3.2) and (3.3) follows

$$\| u_0(x) - u_\varepsilon(x) \|^p_{L_p(0,2\alpha_\varepsilon)} \approx 2 \| u_0(x) - u_\varepsilon(x) \|^p_{L_p(0,\alpha_\varepsilon)} \approx 2^p T_0^{1-p} |\varepsilon \ln \varepsilon|^p, \quad p \geqslant 1 \tag{3.5}$$

When $x = 2\alpha_\varepsilon$ we find from (2.1) and (3.3)

$$u_0(x) - u_\varepsilon(x) = -\beta_\varepsilon + J(\varepsilon, q_\varepsilon, q_\varepsilon, 0) + J(\varepsilon, -w_\varepsilon, 0, v_\varepsilon(x)), \; 2\alpha_\varepsilon \leqslant x \leqslant 2\alpha_\varepsilon + \beta_\varepsilon \tag{3.6}$$

Utilizing the variable substitution given in (3.3) and integrating by parts we obtain, with (3.2) taken into account [11]

$$\int_{2\alpha_\varepsilon}^{2\alpha_\varepsilon+\beta_\varepsilon} J^p(\varepsilon, -w_\varepsilon, 0, v_\varepsilon(x))\,dx \approx \left(\frac{\varepsilon}{2} w_\varepsilon\right)^{1/2(p+1)} \lim_{\varepsilon\to 0} \frac{\partial}{\partial r_\varepsilon} \left(\frac{1}{p+1} K^{p+1}(r_\varepsilon, 1) - \right.$$

$$\left. - 2p \int_0^1 K^{p-1}(r_\varepsilon, x)\,dx \right) r_\varepsilon \approx \Theta(T_0, p)\,\varepsilon, \; r_\varepsilon = w_\varepsilon^{-1}$$

$$K(r, x) = \int_0^x \left(r \ln \frac{r+\eta}{r+1} + 1 - \eta \right)^{-1/2} d\eta \tag{3.7}$$

$$\Theta(T_0, p) = 2^{1-p} T_0^{p-1} (1/p + \frac{1}{p+1} F(1,1,p+2,-1))$$

where $F(a, b, c, z)$ is the Gauss hypergeometric function. Using the Minkowski inequality we obtain from (3.2), (3.6), (2.2) and (3.7)

$$\| u_0(x) - u_\varepsilon(x) \|^p_{L_p(2\alpha_\varepsilon, 2\alpha_\varepsilon+\beta_\varepsilon)} \approx \Theta(T_0, p)\,\varepsilon$$

$$\| u_0(x) - u_\varepsilon(x) \|^p_{L_p(2\alpha_\varepsilon+\beta_\varepsilon, T_0)} = \int_{2\alpha_\varepsilon+\beta_\varepsilon}^{T_0} | u^- - u_\varepsilon(0) - J(\varepsilon, -w_\varepsilon, 0, v_\varepsilon(x)) |^p dx \approx$$

$$\approx 2T_0^{p-1} |\varepsilon \ln \varepsilon| \tag{3.8}$$

Then from (3.5) follows

$$\| U_0(y, t, u^+) - U_\varepsilon(y, t, C_\varepsilon) \|_{L_p(\Omega)} \approx 2^{1/p} T_0^{(p-1)/p} H^{1/p} |\varepsilon \ln \varepsilon|^{1/p}$$

$$\Omega = D_{k,H} = \{(y, t) \mid kT_0 < y - ct < (k+1) T_0, \; 0 < t < H\} \tag{3.9}$$

$$k = 0, \pm 1, \pm 2, \ldots; \; p \geqslant 1, \; H > 0$$

When (3.5) and (3.8) are used, the principal term of the asymptotics in (3.9) can also be computed for regions with a more complicated boundary. From (2.2) and (3.5) it follows that we have $p = 1$ in the right hand side of (3.9), if the boundary Ω contains no points of discontinuity of $U_0(y, t)$.

4. Convergence when the amplitudes of u_ε and u_0 are equal.

From (1.6) - (1.8) it follows that when

$$\Phi = A, \; A(U_\varepsilon(y,t)) = \max_{-\infty<y,t<\infty} U_\varepsilon(y,t) - \min_{-\infty<y,t<\infty} U_\varepsilon(y,t), \; \varepsilon \geqslant 0$$

Eq. (1.4) has a unique solution in M_ε for any $U_0(y, t, u^+) \in M_0$ and any $0 < \varepsilon < \infty$.

Therefore the amplitude functional A has the properties 1 and 2 of Sect. 1.

We now fix arbitrarily $U_0(y, t) \in M_0$ and denote by C^ε the value of the parameter C appearing in the function $U_\varepsilon(y, t, C) \in M_\varepsilon$ which satisfies (1.4) with $\Phi = A$ (Equation connecting C^ε and u^+ is not given here). We also denote (see (1.8))

$$\alpha^\varepsilon = {}^1/_2 T^+(\varepsilon, C^\varepsilon), \quad \psi(\varepsilon, z, w, \gamma) = (1 - zw)^\gamma [2\varepsilon f(zw) + (c - u^+)^2]^{-1/2}$$

$$I(\varepsilon, w, \gamma) = w \int_0^1 \psi(\varepsilon, z, w, \gamma)\, dz, \; u^\varepsilon(x) = u_\varepsilon(x, C^\varepsilon), \; v^\varepsilon(x) =$$

$$= v_\varepsilon(x, C^\varepsilon), \; q^\varepsilon = q_\varepsilon(C^\varepsilon), \; A_0 = A(U_0(y, t, u^+))$$

$$U^\varepsilon(y, t) = U_\varepsilon(y, t, C^\varepsilon), \; a = {}^1/_2 A_0 = c - u^+ = u^- - c$$

When $\Phi = A$ we have from (1.6), (1.7) and (1.4)

$$\ln C^\varepsilon = {}^1/_8 \varepsilon^{-1} A_0^2, \; 2\varepsilon f(q^\varepsilon) + a^2 = 0, \; 2\alpha^\varepsilon - T_0 = 2\varepsilon I(\varepsilon, q^\varepsilon, 0) \to 0 \qquad (4.1)$$

$$\text{for } \varepsilon \to 0$$

Taking (2.1) into account we find that the inclined part ($v^\varepsilon(x) \geqslant 0$) of the profile u_ε converges uniformly in $x \in [0, T_0]$ to the inclined part of the profile u_0

$$\|U_0(y, t) - U^\varepsilon(y, t)\|_{C(\Pi^\delta)} = u_0(T_0) - u^\varepsilon(T_0) \to 0 \;\text{ for } \varepsilon \to 0 \qquad (4.2)$$

$$\Pi^\delta = \{(y, t) \mid \delta \leqslant y - ct \leqslant T_0 - \delta\}, \qquad 0 \leqslant \delta < {}^1/_2 T_0$$

Let $l(\varepsilon, u)$ be the time in which the representative point of (1.6) moves along $L(\varepsilon, C^\varepsilon)$ from the point $(u^-, 0)$ to the point $(u, -z^\varepsilon(u))$, with $v^\varepsilon(x) \leqslant 0$, where $v = -z^\varepsilon(u)$ defines the ordinate v of the cycle $L(\varepsilon, C^\varepsilon)$ on the uv-phase plane as a function of the abscissa u, with $v \leqslant 0$ and $u^+ \leqslant u \leqslant u^-$.

To establish that the functional A has the Property 3 of Sect. 1, it is sufficient by virtue of (4.2) to show that

$$\|2\alpha^\varepsilon + l(\varepsilon, u) - T_0\|_{C[u^+, u^-]} \approx 4A_0^{-1}|\varepsilon \ln \varepsilon| \qquad (4.3)$$

From (1.6) and (1.7) we have

$$l(\varepsilon, u) = -\int_{u^-}^{u} (z^\varepsilon(u))^{-1} du, \qquad u^+ \leqslant u \leqslant u^-$$

$$du = -\varepsilon v \psi(\varepsilon, 1, v, -1)\, dv, \quad z^\varepsilon(c + \sqrt{2\varepsilon f(v) + a^2}) = -v \quad \text{for } v \leqslant 0 \qquad (4.4)$$

$$c \leqslant u \leqslant c + a$$

$$\varepsilon = \frac{{}^1/_8 A_0^2 - {}^1/_2 (u - c)^2}{z^\varepsilon(u) - \ln(1 + z^\varepsilon(u))}, \qquad \varepsilon z^\varepsilon(u) \to \frac{1}{8} A_0^2 - \frac{1}{2}(u - c)^2 \quad \text{for } \varepsilon \to 0$$

which yield [11]

$$l(\varepsilon, u) = \left(\frac{1}{2}\varepsilon z^\varepsilon(u)\right)^{1/2} \int_0^1 (1 + xz^\varepsilon(u))^{-1} \left\{\frac{1}{z^\varepsilon(u)}[f(-xz^\varepsilon(u)) - f(-z^\varepsilon(u))] + \right.$$

$$\left. + \frac{(u - c)^2}{2\varepsilon z^\varepsilon(u)}\right\}^{1/2} dx \approx 2A_0^{-1}|\varepsilon \ln \varepsilon|, \quad c \leqslant u < u^-$$

By the symmetry of the trajectories of (1.6) we have

$$l(\varepsilon, u) = 2l(\varepsilon, c) - l(\varepsilon, 2c - u), \qquad u^+ \leqslant u \leqslant c$$

$$l(\varepsilon, u) \approx 4A_0^{-1}|\varepsilon \ln \varepsilon|, \; u = u^+; \; l(\varepsilon, u) \approx 2A_0^{-1}|\varepsilon \ln \varepsilon|, \; u^+ < u < u^- \qquad (4.5)$$

and using the relations

$$I(\varepsilon, v^{\varepsilon}(x), 0) \approx I(\varepsilon, q^{\varepsilon}, 0) \approx 2A_0^{-1}, \qquad x \in (0, T_0) \tag{4.6}$$

from (4.1) and (4.5) we obtain (4.3).

The relations (4.2) and (4.3) describe the manner in which the RW solution u_ε with a smooth profile $u_\varepsilon(x)$ converges, when $\varepsilon \to 0$ to the RW solution u_0 with an acute-angled profile $u_0(x)$. This is shown graphically in Fig. 2, where the thickness of the lines representing the profiles $u_\varepsilon(x)$ increases with increasing ε

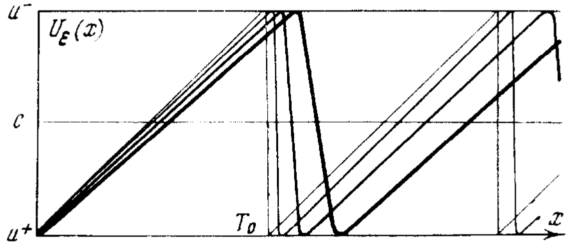

Fig. 2.

5. The rate of convergence when the amplitudes are equal.

Relations (4.3) and (4.5) characterize the rate of convergence of the descending part $(v^{\varepsilon}(x) \leqslant 0)$ of the profile u_ε to the vertical front u_0 in the metric uniform with respect to the ordinate u of the leading front $\{(x, u) \mid x = T_0, \quad u^+ \leqslant u \leqslant u^-\}$ approached by u_0. The last relation in (4.4) defines the rate at which the slope of the descending part of the profile u_ε tends to the vertical slope of the front u_0 (Fig. 2).

Let us find the rate at which the ascending part $(v^{\varepsilon}(x) \geqslant 0)$ of the profile u_ε tends to the ascending part of the profile u_0. From (1.6) and (1.7) we have, for any $\varepsilon > 0$

$$1/_{\varepsilon}(u_0(x) - u_\varepsilon(x)) = \begin{cases} 0, & x = 0, \\ I(\varepsilon, v^{\varepsilon}(x), 0), & x \in (0, \alpha^{\varepsilon}) \\ 2I(\varepsilon, q^{\varepsilon}, 0) - I(\varepsilon, v^{\varepsilon}(x), 0), & x \in [\alpha^{\varepsilon}, T_0] \end{cases} \tag{5.1}$$

Setting in (1.7) $u = u^{\varepsilon}(T_0)$, $v = v^{\varepsilon}(T_0)$ and $C = C^{\varepsilon}$, in (5.1) $x = T_0$ and taking into account (1.8) we eliminate $u^{\varepsilon}(T_0) - u^-$ from the resulting set of two equations. Passing now to the limit with $\varepsilon \to 0$ and making use of (4.2), (4.4) and (4.6), we obtain

$$v^{\varepsilon}(T_0) \approx 1 - e^{-2}, \quad u_0(x) - u^{\varepsilon}(x) \approx 2A_0^{-1}\varepsilon, \quad x \in (0, T_0)$$

$$u_0(T_0) - u^{\varepsilon}(T_0) = \| U_0(y, t) - U_\varepsilon(y, t) \|_{C(\Pi^{\bullet})} \approx 2(1 + e^{-2}) A_0^{-1}\varepsilon \tag{5.2}$$

Writing relations analogous to (5.1) for $x > T_0$, we can find the average rate of convergence of $u_\varepsilon \to u_0$ over an arbitrary region $D_{k,H}$ with $\Phi = A$ (see (3.9))

$$\| U_0(y, t) - U^{\varepsilon}(y, t) \|_{L_p(D_{k,H})} \approx H^{1/p} A_0^{(p-1)/p} \rho(p, k) \, | \varepsilon \ln \varepsilon |^{1/p}$$

$$\rho(p, k) = O(k^{1/p}) \qquad \text{for } k \to \infty, \ p \geqslant 1 \tag{5.3}$$

We note that when a period functional is chosen as Φ (see Sect. 1), the convergence $u_\varepsilon \to u_0$ uniform in yt ceases to hold even in a strip whose width is commensurate with the period u_0 (see (2.3) and (5.2)). When $\Phi = T$ however, the expansion of the region of approximation is not accompanied by decrease in the rate of convergence (5.3).

In the manner analogous to (5.1) and (5.2) we can find the rates at which the derivatives of u_ε converge to the derivatives o

$$\left\| \frac{\partial U_0(y,t)}{\partial y} - \frac{\partial U^\varepsilon(y,t)}{\partial y} \right\|_{L_p(D_{0,H})} = \frac{1}{c} \left\| \frac{\partial U_0(y,t)}{\partial t} - \frac{\partial U^\varepsilon(y,t)}{\partial t} \right\|_{L_p(D_{0,H})} \approx$$

$$\approx \{\varepsilon H [2I(\varepsilon, q^\varepsilon, p-1) - I(\varepsilon, v^\varepsilon(T_0), p-1]\} \approx \left[\frac{2}{p} H A_0^{-1}(1+e^{-2p})\right]^{1/p} \varepsilon^{1/p}$$

Assuming that $2\alpha^\varepsilon > T_0$ (see (4.1)) we obtain from (1.7), (2.1) and (5.2)

$$\left\| \frac{\partial U_0(y,t)}{\partial y} - \frac{\partial U^\varepsilon(y,t)}{\partial y} \right\|_{C(\Pi^\delta)} = \frac{1}{c} \left\| \frac{\partial U_0(y,t)}{\partial t} - \frac{\partial U^\varepsilon(y,t)}{\partial t} \right\|_{C(\Pi^\delta)} =$$

$$= v_0(\delta) - v^\varepsilon(\delta) \approx \exp\left\{(A_0 - 2\delta) A_0^{-1} - \frac{\delta(A_0-\delta)}{2\varepsilon}\right\}, \quad 0 < \delta < \frac{1}{2}(u^- - u^+)$$

By virtue of the periodic character of the solutions considered, it follows from the previous discussion that for each $U_0(y, t) \in M_0$ for any bounded region Ω of the yt-plane and for any $\sigma > 0$ such value of the viscosity $\varepsilon(\sigma, \Omega, U_0)$ can be chosen (for both, $\Phi = T$ and $\Phi = A$) in the condition (1.4)) that when $0 < \varepsilon < \varepsilon(\sigma, \Omega, U_0)$, the flattened part of the integral surface $u = U_0(y, t)$ is found in the σ-neighborhood of the flattened part $(\partial U_0(y, t) / \partial y > 0)$ of the integral surface $u = U_\varepsilon(y, t)$, in the ytu-space, and the steep part $(\partial U_\varepsilon(y, t) / \partial y < 0)$ of the integral surface u_ε is found in the σ-neighborhood of each vertical front $\{(y, t, u) | y - ct = x_i, u^+ \leqslant u \leqslant u^-\}$ u_0 situated in Ω. When $\Phi = T$, the quantity $\varepsilon(\sigma, \Omega, U_0)$ is independent of Ω.

Other functionals exist which have the properties 1 to 3 of Sect. 1 and remain constant with $\varepsilon \to 0$ over such sequences $u_\varepsilon \to u_0$ on which the functuonals T and A are not constant. In particular, T^+ is such a functional defined as the length of this part of the wave base, above which the inclined $(\partial v_\varepsilon(x, C) / dx > 0)$ part of its profile appears. The functional T^+ can be obtained from the second formula of (2.1).

6. Limiting passage on the phase plane. Let us denote

$$rs = \{(u, v) | u^+ \leqslant u \leqslant u^-, \ v = v_0(u) = 1\}$$

$$[r, -\infty) = \{(u, v) | u = u^+, -\infty < v \leqslant 1\}, \ [s, -\infty) = \{(u, v) | u = u^-, -\infty < v \leqslant 1\}, \ Z = rs \cup [r, -\infty) \cup [s, -\infty)$$

Let $\rho = \rho_\varepsilon(\theta)$ be the equation written in the polar coordinates $u - c = \rho\cos\theta$, $v = \rho\sin\theta$ of the phase trajectory $L(\varepsilon, C^\varepsilon)$ of the profile $u_\varepsilon(x, C^\varepsilon)$ defined by the condition (1.4) with $\Phi = A$ on an arbitrarily fixed $u_0 \in M_0$ whose profile is $u_0(x, u^+)$, $-\infty < u^+ < c$. Let also $v = v_+^\varepsilon(u)$ be the equation, in Cartesian coordinates (u, v) on the phase plane, of that part of $L(\varepsilon, C^\varepsilon)$ which lies on the half-plane $v \geqslant 0$. In addition we denote $\rho_0 = (a^2 + 1)^{1/2}$, $\theta^* = \operatorname{arctg} a^{-1}$, $\rho_\varepsilon = \rho_\varepsilon(\theta^*)$, $\eta_\varepsilon = \rho_\varepsilon \sin\theta^*$ and $d_\varepsilon(\theta)$ the distance between the point $(\rho_\varepsilon(\theta), \theta) \in L(\varepsilon, C^\varepsilon)$ and Z.

From (1.6) and (1.7) we obtain the following expression for the deviation between the phase trajectories of the profiles $u_\varepsilon(x, C^\varepsilon)$ and $u_0(x, u^+)$

$$\| v_0(u) - v_+^\varepsilon(u) \|_{C[u^+ + \delta, u^- - \delta]} \approx \exp\left\{-\frac{\delta(2a-\delta)}{2\varepsilon}\right\} \quad (0 < \delta < 2a)$$

$$u^\varepsilon(v) - u^+ \approx -\frac{f(v)}{2a}\varepsilon, \quad u^- - u_+^\varepsilon(v) \approx -\frac{f(v)}{2a}\varepsilon$$

$$u_\pm^\varepsilon(v) = c \pm \psi^{-1}(\varepsilon, 1, v, 0) \tag{6.1}$$

The right hand part of the last relation defines the abscissa u of the phase trajectory $L^{\cdot}(\varepsilon, C^{\varepsilon})$ of the profile $u_{\varepsilon}(x, C^{\varepsilon})$ as a function $u = u_{\pm}^{\varepsilon}(v)$ of the ordinate, the plus sign referring to the right branch $(u > c)$ and the minus sign to the left branch $(u < < c)$. Since δ can be made arbitrarily small, from (6.1) it follows that for arbitrarilly small $\sigma > 0$ and large N there exists $\varepsilon(\sigma, N)$ such that the phase trajectories $L(\varepsilon, C^{\varepsilon})$ of all the profiles $u_{\varepsilon}(x, C^{\varepsilon})$, $0 < \varepsilon < \varepsilon\ (\sigma, N)$ in the half-plane $v \geqslant -N$ do not leave the σ-neighborhood of Z. In particular,

$$\rho_0 - \rho_\varepsilon \to 0, \quad \eta_\varepsilon \to 1 \quad \text{for} \quad \varepsilon \to 0 \tag{6.2}$$

Let us find the deviation of $L(\varepsilon, C^{\varepsilon})$ from Z near the points $(u^+, 1)$ and $(u^-, 1)$ on the uv-plane.

Taking into account the sign of $dv_{+}^{\varepsilon}(u) / du$ determined by (1.6), we have

$$\max_{0 \leqslant \theta \leqslant \pi} d_\varepsilon(\theta) = d_\varepsilon(\theta^*) = (\rho_0 - \rho_\varepsilon) \max\{a, 1\} \rho_0^{-1} \tag{6.3}$$

Rewriting (1.7) in polar coordinates in view of (6.2), we find

$$\rho_\varepsilon - \rho_0 = 2(\rho_0 + \rho_\varepsilon)^{-1} \cos^{-2}\theta^* [\ln(1 - \eta_\varepsilon) + \eta_\varepsilon] \varepsilon \approx \rho_0^{-1} \cos^{-2}\theta^* \ln(1 - \eta_\varepsilon)\varepsilon \tag{6.4}$$

Further, differentiating with respect to ε (1.7) written in the polar coordinates for the point $(\rho_\varepsilon, \theta^*)$, we can express $d\rho_\varepsilon / d\varepsilon$ in terms of η_ε, ρ_ε, ε and θ^*. Utilizing also the expression for ε obtained by going to the polar coordinates in the right-hand side of the third relation of (4.4) at the point $(\rho_\varepsilon, \theta^*)$, using the l'Hospital rule, we obtain

$$\lim_{\varepsilon \to 0} \ln(1 - \eta_\varepsilon) \ln^{-1}\varepsilon = -\sin\theta^* \lim_{\varepsilon \to 0} d\rho_\varepsilon / d\varepsilon\, (1 - \eta_\varepsilon)^{-1} \varepsilon =$$

$$= -{}^1\!/_2 a^2 \lim_{\eta_\varepsilon \to 1} \{(\eta_\varepsilon^2 - 1)[a^2\eta_\varepsilon(1 - \eta_\varepsilon) + {}^1\!/_2 a^2 \eta_\varepsilon(\eta_\varepsilon^2 - 1)(\ln(1 - \eta_\varepsilon) + \eta_\varepsilon)^{-1}]^{-1}\} = 1 \tag{6.5}$$

Now (6.3) and (6.4) yield

$$\max_{0 \leqslant \theta \leqslant \pi} d_\varepsilon(\theta) = d_\varepsilon(\theta^*) \approx a^{-2} \max\{a, 1\} |\varepsilon \ln \varepsilon| \tag{6.6}$$

The part of the trajectory $L(\varepsilon, C^{\varepsilon})$ lying between the points whose polar coordinates are $(a, 0)$ and $(\rho_\varepsilon, \theta^*)$ (its nearness to the segment $\{(u, v) | u = u^-, 0 \leqslant v \leqslant 1\}$ of the ray $[s, -\infty)$) characterized by (6.6)), is traversed by the point representing the system (1.6) in the "time" (see (4.1) and (6.5))

$$I(\varepsilon, \eta_\varepsilon, -1) = \varepsilon I(\varepsilon, \eta_\varepsilon, 0) - \psi^{-1}(\varepsilon, \eta_\varepsilon, 1, 0) + a \to 0 \quad \text{for} \quad \varepsilon \to 0 \tag{6.7}$$

In accordance with the terminology of the theory of relaxation oscillations [13, 14], we shall call the line Z defined by (6.1) and (6.6) the discontinuous trajectory of the degenerate (when $\varepsilon = 0$) system (1.6) corresponding to the profile $u_0(x, u^+)$

From (6.1) and (6.7) it follows that for sufficiently small ε a segment of rs whose length can differ as little as we like from 2a, can be found in an arbitrarily narrow neighborhood of the segment of $L(\varepsilon, C^{\varepsilon})$ contained between the points whose polar coordinates are $(\rho_\varepsilon, \theta^*)$ and $(\rho_\varepsilon, \pi - \theta^*)$ lying in the half-plane $v \geqslant 0$. As follows from (4.5) and (6.7), the segment of the trajectory $L(\varepsilon, C^{\varepsilon})$ lying close to rs, is traversed by the point representing (1.6) in a finite "time" which tends to T_0 when $\varepsilon \to 0$.

Continuing to se the terminology of [13 and 14] we shall call the segment rs of the discontinuous trajectory, the segment of slow motions.

In an arbitrarily narrow neighborhood of the remaining part $(-\pi - \theta^* < \theta < \theta^*)$ of the trajectory $L(\varepsilon, C^{\varepsilon})$ we find segments of the rays $[r, -\infty)$ and $[s, -\infty)$ which can be made arbitrarily long by choosing a sufficiently small ε. From (4.5) and (6.7) it follows that this part of L (ε, C^ε) is traversed by the point representing (1.6) in "time" which becomes vanishingly short when $\varepsilon \to 0$. For this reason we shall call $[r, -\infty) \cup \cup [s, -\infty)$ the segment of rapid motions.

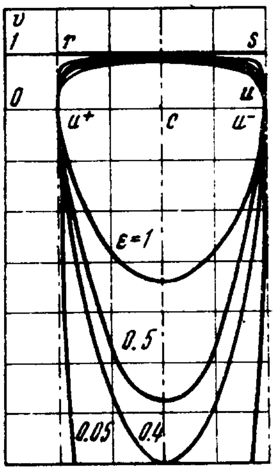

Fig. 3.

Figure 3 illustrates the convergence determined by (6.1), (6.2) and (6.6) of the phase trajectories L (ε C^ε) of the profiles u^ε (x, C^ε) fixed by the condition (1.4) with $\Phi = = A$, to the discontinuous phase trajectory Z of the profile u_0 (x, u^+) of an arbitrary RW solution $u_0 \in M_0$.

Thus the discontinuity trajectory consists, as in the case of relaxation oscillations, of alternating segments of slow and rapid motion. The latter segment, in turn, consists of a section of instantaneous leaving the point s ("break-off" point) to $-\infty$ along the ray $[s, -\infty)$ and a section of instantaneous return ("drop" point) from $-\infty$ to the point r along the ray $[r, -\infty)$ [13, 14]. The segment of rapid motions connects (through the point at infinity) points at which the variable u assumes its maximum and minimum value, the points themselves situated on the segments rs of slow motions representing the inclined part of the profile u_0 (x, u^+) on the uv-plane. Thus the segment of rapid motions on the uv phase plane plays the part of the vertical section of the profile u_0 (x, u^+) when the latter is mapped onto the xu plane. We must therefore treat the segment of rapid motions as a representation of the vertical part of the profile u_0 (x, u^+) on the uv phase plane.

Let us note the difference between the passage to the limit considered here and the passage to the limit encountered in the theory of relaxation oscillations. The representative point of the degenerate system moving on the discontinuous trajectory along the segment of rapid motions changes its direction of motion without reaching the segment of slow motions and arrives at the point of return r, moving in the direction opposite to that at s at the instant of break-off.

The coordinates of the break-off and the drop points make the right-hand side of the equation with a parameter in (1.6) vanish, just as in [13, 14], but these points lie on a straight line orthogonal to those along which rapid motion takes place. In the case of the relaxation oscillations, one of the defining conditions at the discontinuity is that the consecutive break-off and drop points lie on the straight line of rapid motions. On the other hand, in the present case the conditions holding at the discontinuity are analogous to the hydrodynamic conditions of conservation and dissipation corresponding to Eq. (1.2). Moreover, the direction of the break-off is not tangential but orthogonal to the trajectory of slow motions.

We also note that in the present case no value of $\varepsilon > 0$ exists for which a neighborhood of the discontinuous trajectory could be found narrow enough to contain a single periodic solution of the system with $\varepsilon > 0$ [14]. On the contrary, in an arbitrarily narrow neighborhood of the discontinuous trajectory we find, for any $\varepsilon > 0$, a continuum of closed trajectories of the system (1.6), (1.7) with $\varepsilon > 0$.

Distinctions from the theory of relaxation oscillations result from the fact that the line

$$\{(u,\ v)\mid v=1,\ -\infty<u<\infty\}\cup\{(u,\ v)\mid u=c,\ -\infty<v<\infty\}$$

lying on the uv phase plane of the system (1.6) and composed of the states of equilibrium of the equation containing ε (provided that the coordinate entering this equation without ε accompanying the derivative is regarded as a parameter varying over the whole numerical axis) is not a simple curve as is assumed in [13, 14]. This line branches at the point $(c,\ 1)$, thus the state of equilibrium $v=1$ of the first equation of (1.6) with fixed $u=c$ is not isolated.

Characteristically, the merging of the stable and unstable states of equilibrium of (1.6) containing ε taking place at $(c,\ 1)$ does not lead to the break-off from the trajectory of slow motions approaching the point of merger, as it happens in the case of the relaxation oscillations.

Contrary to (6.1), the closest approach made by the trajectory $L(\varepsilon,\ C^{\varepsilon})$ at this point is, for each $\varepsilon>0$, to the segment rs of slow motions. Irrespective of the fact that rs to the right of $(c,\ 1)$ (Fig. 3) consists of unstable states of equilibrium, the slow motion along rs continues and the length of the unstable segment of slow motions traversed without break-off is equal to half of the amplitude of the approached u_0. Since the latter is arbitrary, this length can be made as large as we please (1.8).

Although the segment of the trajectory of slow motions lying to the right of $(c,\ 1)$ is unstable if we take into account the character of the component states of equilibrium, it is stable in the following limiting sense. In an arbitrarily narrow neighborhood of this segment we can find, for all sufficiently small $\varepsilon>0$, as was shown before, a continuum of orbitally stable trajectories of (1.6) which remain within this neighborhood between the beginning $(c,\ 1)$ of this segment and a point arbitrarily near to its end point s.

The passage to the limit on the phase plane when a functional different from A is chosen as Φ in (1.4), can be dealt with in a similar manner.

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ON CONVECTION ONSET IN A SELF-GRAVITATING FLUID SPHERE WITH INTERNAL HEATING

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Proof is given that the loss of stability of the equilibrium state of a self-gravitating fluid filling a rigid sphere having uniformly distributed internal heat sources is accompanied by the onset of a stationary axisymmetric (correct to within an arbitrary rotation) flow which remains stable in the vicinity of the point of stability loss. This flow is numerically defined as a segment of the Liapunov-Schmidt series. The problem of thermal instability of a self-gravitating fluid sphere is associated with various theories and hypotheses of geo- and astro-physics, as well as with the study of fluid behavior in conditions of quasi-weightlessness. Earlier investigations were mainly directed toward the formulation and solution of the linearized problem and finding the limit of instability [1]. Their results were further developed in later publications([2, 3] and others). The method proposed by Chandrasekhar [1] was applied in [4] to the related nonlinear problem. The theory of solution branching of equations of stationary convection [5, 6] is applied below to the study of convection onset in a self-gravitating fluid sphere.

1. Statement of problem. A rigid sphere S or radius a is filled with a viscous incompressible fluid acted upon by a spherically symmetric radial gravitational